MATCHING EXTENSION IN REGULAR GRAPHS

by

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Abstract

This paper deals with extending matchings in regular graphs. There are two main results. The first presents a sufficient condition in terms of cyclic connectivity for extending matchings in regular bipartite graphs. This theorem generalizes an earlier result due to Holton and the author. The second result deals with regular—but not necessarily bipartite—graphs. In this case, it is known that a result analogous to that obtained in the bipartite case is impossible, but a new proof is given of a result of Naddef and Pulleyblank which guarantees that a regular graph with an even number of points which has sufficiently large cyclic connectivity will be bicritical.

1. Introduction and Terminology

All graphs considered are finite, undirected, connected and simple (i.e., they have no loops or parallel lines). Let n and p be positive integers with $n \leq (p-2)/2$ and let G be a graph with p points having a perfect matching. Graph G is said to be n-**extendable** if every matching of size n in G extends to a perfect matching. For a discussion of the role of the concept of n-extendability within the general framework of matching theory in graphs and for an historical résumé of the development of n-extendability, the reader is referred to the book [9] and to [10].

In this paper we continue work begun in [3], [4] and [2].

In Section 2 we generalize a result on matching extension in bipartite graphs obtained in [4]. This result—and the present generalization—involve the concept of *cyclic connectivity.*

A set L of lines in a connected graph G is called a **cyclic cutset** if $G-L$ is disconnected and at least two of the components of $G-L$ contain cycles. If graph G has a cyclic cutset of lines, we define the **cyclic (line) connectivity** of G to be the cardinality of any smallest cyclic cutset in G and denote this number by $c\lambda(G)$. If G has no such set, we shall say that the cyclic connectivity of G is infinite and write $c\lambda(G) = +\infty$. (The reader is warned that some authors prefer to say that when no cyclic line cutset is present, the cyclic connectivity is defined to be 0.)

In addition to cyclic connectivity, we shall refer to other graph parameters such as *regularity, bipartiteness, planarity* and both *point* and *line connectivity.* When various combinations of these parameters are needed in the statements of a number of theorems

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and corollaries, for example, it will be convenient to use juxtaposed abbreviations. For example, "Let graph G be 3-regular, 4-connected, bipartite, planar and have an even number of points" might be shortened to "Let graph G be $3R4CBPE$".

In Section 3, we study possible analogs of the results of Section 2 in the case when the bipartite property is dropped. Although we can say very little about n-extendability for $n \geq 2$ in this situation, we do obtain a new proof of a theorem of Naddef and Pulleyblank [12] which guarantees that certain families of regular graphs are *bicritical*. A graph G is **bicritical** if $G - u - v$ has a perfect matching for every choice of two distinct points u and v in $V(G)$.

2. The bipartite case

In this section we shall generalize somewhat a result on extending matchings in regular bipartite graphs first obtained in [4].

Before presenting our main result, we state a lemma the proof of which is trivial and is left to the reader.

Lemma 2.1. Let F be a forest with no isolated points. Suppose the bipartition of $V(F)$ is $A \cup B$, where $|A| = a$, $|B| = b$ and $a > b$. Then F contains a tree with at least 2 endpoints in A. ∎

Now we are prepared to state and prove the main result of this section.

Theorem 2.2. Let n and r be a positive integers with $r \geq n+1$ and suppose graph G is rRB. Then if $c\lambda(G) \geq (n-1)r + 1$, G is n-extendable.

Proof. If $n = 1$, G is 1-extendable by the well-known line coloring theorem of Kōnig [5, 6] and cyclic connectivity does not need to be considered.

So let us assume that $n \geq 2$. Suppose graph G satisfies the hypotheses of this theorem and suppose further that the bipartition of its points is $V(G) = A \cup B$. Now suppose that G is not n-extendable. Then there are n independent lines $e_1 = a_1b_1, \ldots, e_n = a_nb_n$ where each point $a_i \in A$ and $b_i \in B$ and such that $G' = G - a_1 - \cdots - a_n - b_1 - \cdots - b_n$ has no perfect matching. Thus by the well-known theorem of Philip Hall on bipartite matching, we may assume, without loss of generality, that there exists a point set $A_1 \subseteq A$ with $|\Gamma_{G'}(A_1)| < |A_1|$. (Here $\Gamma_{G'}(A_1)$ denotes the set of neighbors of set A_1 in graph G'.) Moreover, since graph G is r-regular and $r \geq n+1$, set A_1 contains no isolates in G'. Let us denote $\Gamma_{G'}(A_1)$ by B_1. Let $G_1 = G[A_1 \cup B_1]$ and let $G_0 = G[A_0 \cup B_0]$ where $A_0 = A - (A_1 \cup \{a_1, \ldots, a_n\})$ and $B_0 = B - (B_1 \cup \{b_1, \ldots, b_n\})$.

Note here at the outset that the pair (A_1, B_1) where $\Gamma_{G'}(A_1) = B_1$, point set A_1 contains no isolates and $|\Gamma_{G'}(A_1)| = |B_1| < |A_1|$ is certainly not necessarily unique. Let us call such a pair (A_1, B_1) a **Hall barrier** in G'. (For more on the barrier concept in general, see [9].)

Note that $B_1 \neq \emptyset$, since $\deg_{G'} v > 0$ for all points $v \in A_1$. Thus $|B_1| \geq 1$ and hence $|A_1| \geq 2$. (Similarly, $B_0 \neq \emptyset$, $|A_0| \geq 1$ and hence $|B_0| \geq 2$.)

For the rest of this proof, let us adopt the following terminology:

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A-1

$$
\begin{aligned}
m &= \text{ the number of lines from } A_1 \text{ to } \{b_1, \ldots, b_n\}, \\
n_0 &= \text{ the number of lines from } B_0 \text{ to } \{a_1, \ldots, a_n\}, \\
n_1 &= \text{ the number of lines from } B_1 \text{ to } A_0, \\
n_2 &= \text{ the number of lines from } B_1 \text{ to } \{a_1, \ldots, a_n\}, \\
n_3 &= \text{ the number of lines from } \{b_1, \ldots, b_n\} \text{ to } A_0, \text{ and} \\
n_4 &= \text{ the number of lines in } G[S],
\end{aligned}
$$

where $S = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}$. (See Figure 2.1 below. Line cut L shown in that figure will be discussed later in this proof.)

Figure 2.1.

Now counting lines in G_1, we have $|A_1|r - m = |B_1|r - n_1 - n_2$, or

$$|A_1|r = |B_1|r - n_1 - n_2 + m. \tag{1}$$

Also $|A_1| \geq |B_1| + 1$, so

$$|A_1|r \geq (|B_1| + 1)r. \tag{2}$$

Combining (1) and (2), we get $(|B_1| + 1)r \leq |A_1|r = |B_1|r - n_1 - n_2 + m$, or

$$n_1 + n_2 \leq m - r. \tag{3}$$

Claim 1. If G_0 is a forest, then $|A_0| + |B_0| \leq 2n + 1$.
Since G_0 is a forest, $|V(G_0)| \geq |E(G_0)| + 1$, and so

$$
\begin{aligned}
2(|V(G_0)|) &= 2(|A_0| + |B_0|) \\
&\geq 2(|E(G_0)| + 1) \\
&= \sum_{v \in V(G_0)} \deg_{G_0} v + 2 \\
&= |A_0|r - n_1 - n_3 + |B_0|r - n_0 + 2
\end{aligned}
$$

or

$$
(r-2)(|A_0| + |B_0|) \leq n_0 + n_1 + n_3 - 2. \tag{4}
$$

Now counting lines out of $\{a_1, \ldots, a_n\}$ in G, we also have

$$
rn = n_0 + n_2 + n_4 \geq n_0 + n_2 + n
$$

so

$$
n_0 + n_2 \leq (r-1)n
$$

and hence

$$
n_0 \leq (r-1)n - n_2, \tag{5}
$$

and similarly, counting lines out of $\{b_1, \ldots, b_n\}$ in G, we also have

$$
n_3 \leq (r-1)n - m. \tag{6}
$$

Thus substituting (5), (3) and (6) into (4), we obtain

$$
\begin{aligned}
(r-2)(|A_0| + |B_0|) &\leq (r-1)n - n_2 + n_1 + n_3 - 2 \\
&= (r-1)n + n_1 + n_2 - 2n_2 + n_3 - 2 \\
&\leq (r-1)n + m - r - 2n_2 + n_3 - 2 \\
&\leq (r-1)n + m - r - 2n_2 + (r-1)n - m - 2 \\
&= 2(r-1)n - r - 2n_2 - 2 \\
&\leq 2(r-1)n - r - 2.
\end{aligned}
$$

Now $r \geq n + 1 \geq 3$, so $r - 2 > 0$. Hence

$$
\begin{aligned}
|A_0| + |B_0| &\leq \frac{2(r-1)n}{(r-2)} - \frac{(r+2)}{(r-2)} \\
&< 2n\frac{r-1}{r-2} - \frac{r-1}{r-2} \\
&= (2n-1)\left(\frac{r-1}{r-2}\right) \\
&= (2n-1)\left(1 + \frac{1}{r-2}\right) \\
&= 2n - 1 + \frac{2n-1}{r-2} \\
&\leq 2n - 1 + \frac{2n-1}{n-1} \\
&= 2n - 1 + \frac{2n-2}{n-1} + \frac{1}{n-1} \\
&= 2n + 1 + \frac{1}{n-1}.
\end{aligned}
$$

But $|A_0| + |B_0|$ is an integer and $n \geq 2$. So it follows that $|A_0| + |B_0| \leq 2n+1$ and Claim 1 is proved.

Claim 2. $n_1 + n_2 + n_3 + n_4 \leq r(n-1)$.
For we have

$$
\begin{aligned}
n_1 + n_2 + n_3 + n_4 &= n_1 + n_2 + n_3 + (rn - m - n_3) \\
&\leq m - r + n_3 + (rn - m - n_3) \\
&= r(n-1).
\end{aligned}
$$

Now among all Hall barriers in G', let (A_1, B_1) be one with the smallest number of points. Without loss of generality, we may assume that $A_1 \subseteq A, B_1 \subseteq B$ and $|A_1| \geq |B_1| + 1 \geq 2$. Let $G_1 = G[A_1 \cup B_1]$. We now define $H_0 = G[A_0 \cup B_0 \cup \{a_1, \ldots, a_n\}]$ and $H_1 = G[A_1 \cup B_1 \cup \{b_1, \ldots, b_n\}]$.

Claim 3. Subgraph H_1 contains a cycle.

Suppose not. Then H_1 is a forest and hence so is G_1. Since G_1 is a Hall barrier, it contains no isolates. Since $|A_1| > |B_1|$, we may apply Lemma 2.1 to conclude that G_1 contains a tree T_1 with at least 2 endpoints in A_1. But since G is r-regular, each of these endpoints is adjacent to all of $\{b_1, \ldots, b_n\}$ and hence, since $n \geq 2$, G_1—and therefore H_1—must contain a 4-cycle, a contradiction.

Claim 4. $|A_0| = |B_0| - 1$.

We will show that $|A_1| = |B_1| + 1$, and the claim will follow. Suppose $|A_1| \geq |B_1| + 2$.

Suppose $\alpha \in A_1$. Furthermore, suppose that every neighbor in B_1 of α is adjacent to some point in $A_1 - \alpha$. Then $(A_1 - \alpha, B_1)$ is a smaller Hall barrier than (A_1, B_1), a contradiction.

So for every $\alpha \in A_1$ there must exist a $\beta \in B_1$ such that α is adjacent to β, but β is adjacent to no other point in A_1. But then we must have a matching of A_1 into B_1 and hence $|A_1| \leq |B_1|$ which is again a contradiction.

Claim 5. H_0 contains a cycle.

Let $\ell = |B_0|$. Then $|A_0| = \ell - 1$ by Claim 4. Now suppose H_0 does not contain a cycle. Then H_0 is a forest (and hence so is G_0.) Then

$$\begin{aligned}|V(H_0)| &= |A_0| + |B_0| + n \\ &= 2\ell - 1 + n \\ &\geq |E(H_0)| + 1 \\ &= (n + \ell - 1)r - (n_1 + n_2 + n_3 + n_4) + 1 \\ &\geq (n + \ell - 1)r - (n - 1)r + 1 \\ &= \ell r + 1.\end{aligned}$$

So in particular, $\ell r + 1 \leq 2\ell + n - 1$ and hence

$$\ell r \leq 2\ell + n - 2. \tag{7}$$

But then again using the fact that $r \geq n + 1$ and inequality (7), we have $\ell(n + 1) \leq \ell r \leq 2\ell + n - 2$ and so

$$\ell n \leq \ell + n - 2. \tag{8}$$

Now from Claim 1, we have $2\ell - 1 \leq 2n + 1$ and hence $2\ell \leq 2n + 2$ or $\ell \leq n + 1$. Substituting this into inequality (8), we obtain $\ell n \leq n + 1 + n - 2 = 2n - 1$. But $\ell \geq 2$, so $2n \leq \ell n \leq 2n - 1$, a contradiction, and Claim 5 is proved.

Now by Claims 3 and 5, if L is the set of lines counted by $n_1 + n_2 + n_3 + n_4$ (see Figure 2.1), then L is a *cyclic* line cut and hence by Claim 2 $c\lambda(G) \leq |L| \leq r(n - 1)$, contradicting the hypothesis and completing the proof of the theorem. ∎

One version of König's Line Coloring Theorem [5, 6] (also see Chapter 1 of [9]) can be phrased as follows: every regular bipartite graph has a perfect matching. It then immediately follows that every regular bipartite graph decomposes into a union of line-disjoint perfect matchings. So, in particular, every line of a regular bipartite graph lies in some perfect matching; i.e., every regular bigraph is 1-extendable. This result can be compared with the following corollary to the preceding theorem.

Corollary 2.3. If $r \geq 3$ and G is an r-regular bipartite graph with $c\lambda(G) \geq r + 1$, then G is 2-extendable.

Proof. Let $n = 2$ in Theorem 2.2. ∎

The bound $c\lambda(G) = r + 1$ in the preceding corollary is sharp as may be seen by the following infinite family.

Let $\mathcal{G}_1 = \{G_r^1\}_{r=3}^{\infty}$ be the infinite family shown in Figure 2.2. (As usual, the + sign between two sets of points denotes the join operation; i.e., each point of the left hand set is joined by a line to each point of the right hand set.)

Figure 2.2. The infinite family $\mathcal{G}_1$

Note that $|V(G_r^1)| = 4r - 2$ and that $\kappa(G_r^1) = c\lambda(G_r^1) = r$. Note also that G_r^1 is not 2-extendable since $\{e_1, e_2\}$ does not extend to a perfect matching.

Corollary 2.4. If n is any positive integer, if graph G is $(n+1)$-regular and bipartite and if $c\lambda(G) \geq n^2$, then G is n-extendable.

Proof. Let $r = n + 1$ in Theorem 2.2. ∎

Corollary 2.5. (=Theorem 3.2 of [4]) If n is any positive integer, if graph G is $(n+1)R(n+1)CB$ and if $c\lambda(G) \geq n^2$, then G is n-extendable. ∎

Several remarks are in order here. First note that Corollary 2.4 strengthens the main theorem of [4] in that the assumption that G be $(k+1)$-connected has been dropped from the hypotheses. Actually, the fact that G is $(n+1)$-connected now follows from Theorem 2.2 or Corollary 2.4 by applying Theorem 3.2 of [13].

At this point, however, we hasten to point out that the proof of Theorem 2.2 is virtually the same as that of Theorem 3.2 of [4] in most respects. The only differences are that the $(n+1)$-connected assumption has been avoided by a simple regularity argument and a bit more care in counting has been employed. We also wish to point out that if G is 2-regular and bipartite (i.e, if $n = 1$ in the first hypothesis of Corollary 2.4) then G must be an even cycle and by definition, the cyclic connectivity is infinite in this case. Of course even cycles are trivially 1-extendable.

For every $n \geq 2$ the bound of n^2 for $c\lambda(G)$ in both Corollaries 2.4 and 2.5 is sharp. This was demonstrated by Lou and Holton [7; Corollary 6.2] who used probabilistic methods

to show that for every such n there exists a graph G which is $(n+1)R(n+1)CB$, has $c\lambda(G) = n^2 - 1$ and is *not* n-extendable.

Next we look at some consequences of Theorem 2.2 for bipartite *planar* graphs. Of course r-regular planar graphs only exist for $r \leq 5$. Since $r = 1$ implies $G = K_2$ and $r = 2$ implies G must be an even cycle, we shall concern ourselves only with values $r = 3, 4$ and 5. But it is an easy consequence of Euler's formula that there are no 4-regular or 5-regular bipartite *planar* graphs. Moreover, if G is 3-regular bipartite and planar, again using Euler's theorem, it is easy to see that G must have a quadrilateral face and hence $c\lambda(G) \leq 4$. So for regular bipartite planar graphs, Theorem 2.2 tells us only the following.

Corollary 2.6. If graph G is 3RBP and $c\lambda(G) = 4$, then G is 2-extendable. ∎

In a note added in proof to [3] it was remarked that the fact that every G which is 3R3CBP is 2-extendable follows as an immediate corollary of Theorem 3.2 of [4]. Corollary 2.6 strengthens this result by dropping the assumption that G be 3-connected from the list of hypotheses; that is, in just the same way that Corollary 2.4 strengthens Theorem 3.2 of [4].

The requirement that $c\lambda(G) = 4$ in Corollary 2.6 is sharp in the sense that there are graphs G which are 3R3CBP with $c\lambda(G) = 3$, but which are not 2-extendable. Figure 2.3 shows an infinite family $\mathcal{G}_2 = \{G_r^2)\}_{r=2}^{\infty}$ of such graphs. (Note that $|V(G_r^2)| = 6r + 2$.)

Figure 2.3. The infinite family $\mathcal{G}_2$

3. The non-bipartite case

If we consider *non*-bipartite graphs we can expect no nice n-extendability results analogous to those obtained in the preceding section for bipartite graphs. This is immediate from another result of Lou and Holton [7; Corollary 6.3] which states that for any integers $m > 0, k \geq 3$ and $n \geq 2$, there is a cyclically m-line-connected graph which is k-regular and k-connected, but not n-extendable.

However, all is not lost as we can obtain a large family of *bicritical* graphs, provided the cyclic connectivity is sufficiently large. First, however, we present two simple results about cyclic line connectivity for graphs in general.

Lemma 3.1. Suppose mindeg $G = r \geq 3$ and let L be a minimum line cut in G. Then either

(a) L is the star of some point in G or

(b) L is a minimum *cyclic* line cut of G and hence $|L| = \lambda(G) = c\lambda(G) \leq r =$ mindeg G in this case.

Proof. We need only show that $c\lambda(G) \leq \lambda(G)$. Suppose L is a minimum line cut in G and L is not the star of a point. Let C_1 and C_2 be the two components of $G - L$. So each C_i contains at least two points.

Suppose C_1 does not contain a cycle. Then C_1 is a tree with at least two endpoints and hence $|L| \geq 2r-2$. So $2r-2 \leq |L| = \lambda(G) \leq$ mindeg $G = r$. So $r \leq 2$, a contradiction. So C_1 must contain a cycle and similarly, so must C_2. Thus L is a cyclic line cutset and thus $c\lambda(G) \leq |L| = \lambda(G)$. But $\lambda(G) \leq c\lambda(G)$ by definition and the Lemma follows. ∎

We remark that the conc on of this Lemma does not hold if mindeg $G = 2$, for simply let G be any cycle with at least four points.

Corollary 3.2. If mindeg $G = r \geq 3$ and $c\lambda(G) \geq r + 1$, then all minimum line cuts in G are stars and hence $\lambda(G) = r$.

Proof. Let L be any minimum line cut which is not a star. Then $|L| = \lambda(G) = c\lambda(G) \leq r$ by Lemma 3.1, thus contradicting the hypothesis. Thus all minimum line cutsets in G are stars and the conclusion follows. ∎

We note that the following theorem was proved in an essentially equivalent form by Naddef and Pulleyblank [12; Theorem 6] and a closely related result appears in Naddef [11; Theorem II.3]. However, the Naddef-Pulleyblank proof uses results from polyhedral theory, in particular, Edmonds' characterization of the matching polytope. We include a different proof here both for the sake of completeness and because it requires no notions from the theory of polyhedra; only Tutte's classic theorem on perfect matchings.

Theorem 3.3. Let $r \geq 3$ be an integer and let G be a graph which is rRE with $c\lambda(G) \geq r + 1$. Then if G is not bipartite, it is bicritical.

Proof. By Corollary 3.2, we have $\lambda(G) = r$ and hence by [1; Thm 13, p 160] (see also [9; Thm 3.4.3]) G is 1-extendable.

Suppose now that $G' = G - u - v$ has no perfect matching. Thus by Tutte's theorem on perfect matchings, there is a set $S' \subseteq V(G')$ such that $|S'| < c_o(G' - S')$, where $c_o(G' - S')$ denotes the number of odd components of graph $G' - S'$. Thus, since G is even, by parity we have $|S'| \leq c_o(G' - S') - 2 = c_o(G - S) - 2$. So if $S = S' \cup \{u, v\}$, we have $|S| \leq c_o(G - S)$ and since G contains a perfect matching, $|S| = c_o(G - S)$.

Moreover, since G is 1-extendable, $G - S$ has no even components and subgraph $G[S]$ is an independent set. Now if all components of $G - S$ are singletons, G is bipartite, contrary to hypothesis. So, without loss of generality, assume that C_1 is an odd component of $G - S$ with $|V(C_1)| \geq 3$. If C_1 were a tree, it would have at least two endpoints and hence there would be at least $2(r-1) = 2r - 2$ lines joining C_1 to S. Thus if $|S| = s$, since $\lambda(G) = r$ the subgraph $C_1 \cup \cdots \cup C_s$ sends at least $2r - 2 + (s-1)r = sr + r - 2 \geq sr + 1$ lines to S, where we have used the fact that $r \geq 3$.

On the other hand, since G is r-regular and S is independent, S must send exactly sr lines to $C_1 \cup \cdots \cup C_s$. This is a contradiction.

Thus C_1 (and hence all odd components of $G - S$ which contain at least three points) contain cycles.

We now claim that subgraph $H_1 = G[V(G) - V(C_1)]$ also contains a cycle. Suppose not. Then H_1 is a forest on $2s - 1$ points and hence contains no more than $2s - 2$ lines. On the other hand, also because H_1 is a forest, each of $C_2, \ldots, C_s$ is a singleton and each therefore sends exactly r lines to S. In other words, forest H_1 contains exactly $r(s-1)$ lines. Thus $r(s-1) \leq 2s - 2$ and since $s \geq 2$ it follows that $r \leq 2$, a contradiction.

Thus graph H_1 contains a cycle. But then since $c\lambda(G) \geq r+1$, viewed from $C_1 \cup \cdots \cup C_s$, $|E(G)| \geq (r+1) + (s-1)r = sr + 1$, whereas viewed from S, G has exactly sr lines. Thus we have a contradiction and the proof is complete. ∎

Bicritical graphs which are 3-connected are called **bricks**. Bricks play an important role in a theory for the decomposition of graphs in terms of their maximum matchings which has been developed over the past thirty years or so. For a treatment of this theory in depth, the reader is referred to [9] and [8].

Corollary 3.4. Suppose $r \geq 3$ is an integer and G is a graph which is rRE with $c\lambda(G) \geq r+1$. Then if G is not bipartite, G is a brick.

Proof. Immediate. ∎

The bound on the cyclic connectivity in Corollary 3.4 above is sharp for all $r \geq 3$. To see this, we present an infinite family $\mathcal{G}_3 = \{G_r^3\}_{r=3}^{\infty}$ constructed as follows. First suppose r is odd. Let S denote a set of r independent points. Then G_r^3 is the graph on $r^2 + r$ points formed by joining each point of r disjoint copies of the complete graph K_r to the set S by a perfect matching. Now suppose r is even (and hence $r \geq 4$). Let S be as in the odd case above. Form a graph H_j as follows. Let $K_r - pm$ denote the complete graph K_r with a perfect matching deleted. Now join a new point v_j to each point of $K_r - pm$ and denote the resulting graph on $r+1$ points by H_j. To form graph G_r^3, for each $j = 1, \ldots, r$ join H_j to S via a near-perfect matching which matches all points of H_j—except v_j—to S. Note that for r even, graph G_r^3 has $r^2 + 2r$ points

Clearly, each G_r^3 is r-regular and has cyclic line connectivity exactly r. Furthermore, it is easy to see that none of these graphs is bicritical, for if one deletes any two points from S in any of them, the resulting graph cannot have a perfect matching.

It is easy to see that for each $r \geq 3$, if one shrinks the odd components of $G_r^3 - S$ to single points, one obtains the complete bipartite graph $K_{r,r}$. Hence all of the graphs in the family $\mathcal{G}_3$ are non-planar. For the applicable values of r (namely, for $r = 3, 4, 5$) it then makes sense to ask if there are *planar* examples of graphs which are r-regular, have cyclic line connectivity r, but are not bicritical. The answer is "yes" in each case and we present an infinite family for each of the three values for r. For $r = 3, 4$ and 5 respectively, form the infinite families $\{G_j(A)\}_{j=2}^{\infty}$, $\{G_j(B)\}_{j=2}^{\infty}$ and $\{G_j(C)\}_{j=2}^{\infty}$ where A, B and C are the graph fragments shown in Figure 3.1.

The members of all three families are clearly planar. Moreover, the three families are respectively 3-regular, 4-regular and 5-regular with cyclic connectivities 3, 4 and 5. None of these graphs can be bicritical, for note that the removal of any two points in the set $\{u_1, \ldots, u_j, v_1, \ldots, v_j\}$ results in a graph with no perfect matching.

Clearly the member of all three families shown in Figure 3.1 are 3-connected. If G is $rRPE$ for $r = 4$ or 5 and G is 4-connected, then G *is* bicritical (and hence a brick) by Theorem 2.1 of [14].

Finally, we make one last observation with respect to Corollary 3.4. If one adds to the hypotheses of this corollary the condition that G has no quadrilateral faces, then in fact G is not only bicritical, but 2-extendable. (For a proof of this fact, see [3].)

Figure 3.1.

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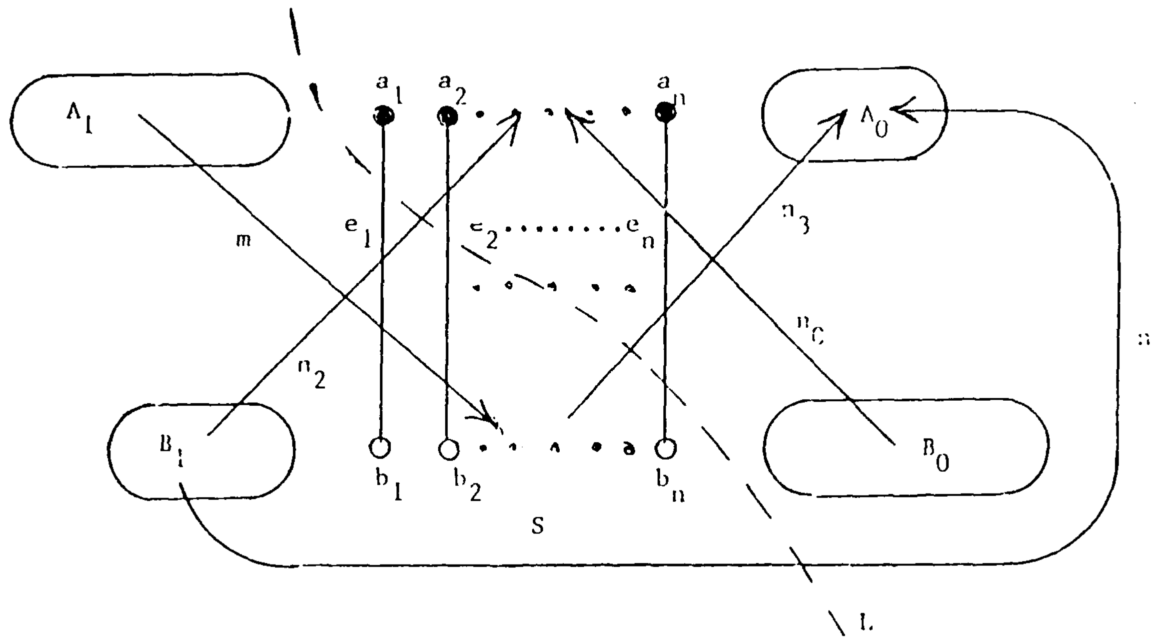

Figure 2.1

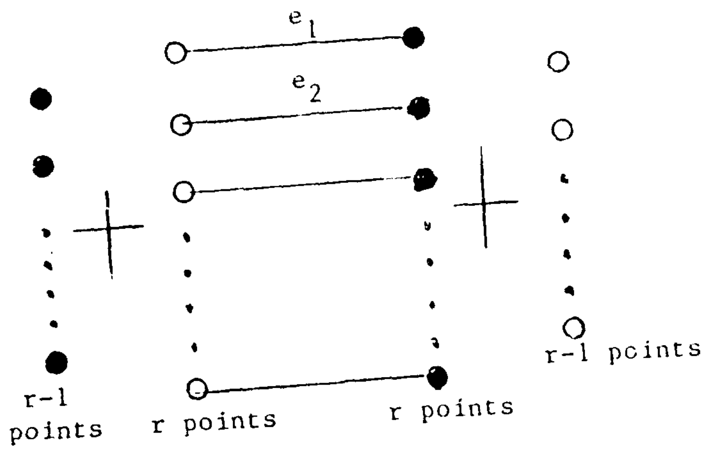

Figure 2.2. The infinite family $\mathcal{B}_1$

G_r^2:

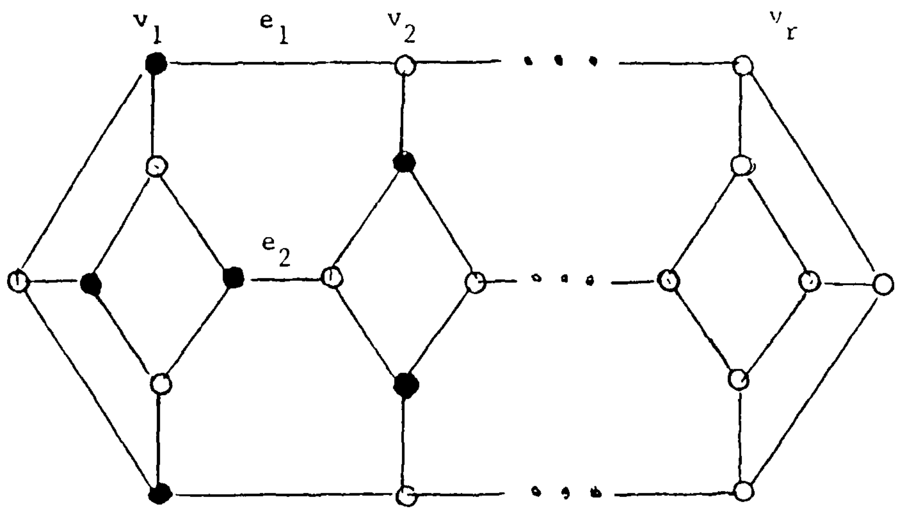

Figure 2.3. The infinite family $\mathcal{G}_2$

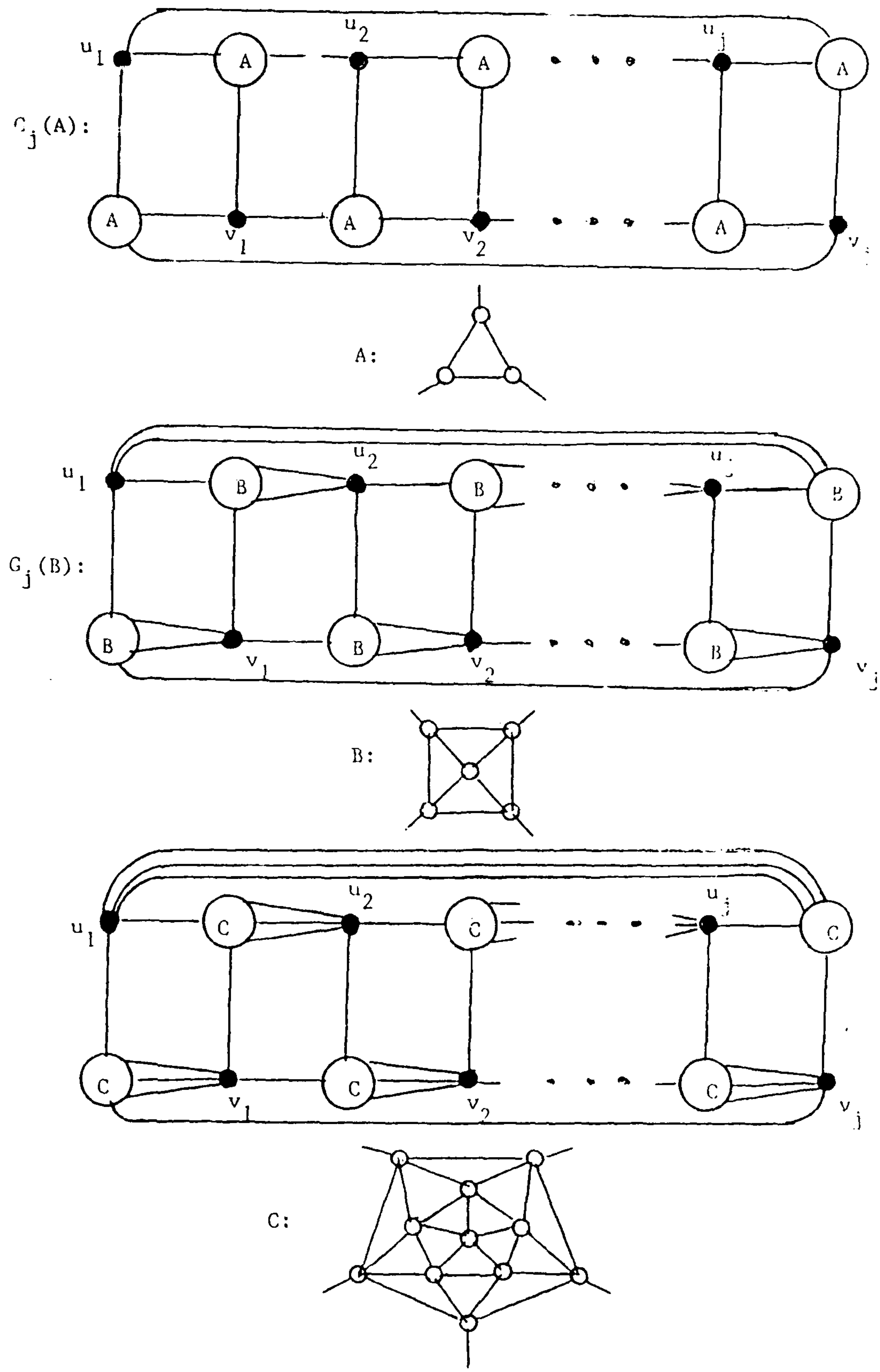

Figure 3.1